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DETERMINING THE NASH EQUILIBRIUM FROM THE REACTION RELATIONS OF THE DECISION MAKERS

DANIEL PATRICK O'CONNORS

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DETERMINING THE NASH EQUILIBRIUM FROM THE REACTION
RELATIONS OF THE DECISION MAKERS

BY

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THESIS

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1. INTRODUCTION

Many of the problems in decision and control theory involve estimation and optimization. Several methods including self-tuning regulators and model reference adaptive schemes are available for estimating and controlling systems with unknown parameters [1],[2],[3]. The theory for the optimization of a single performance index for both deterministic and stochastic systems with known parameters is well-established [4],[5],[6]. The theory for the identification and control of systems with several decision makers, each having different information available and each having his own performance index, is much more difficult [7]. There are several conceptual reasons why the classical theories for systems with single cost functions cannot be easily adjusted to handle multiple cost functions. First, it may not be possible to optimize the multiple objectives simultaneously. Second, the information available to each user is not necessarily the same. These problems do not occur for the single objective case.

1.1. Overview of Multi-User Control Theory

The problem of optimizing multiple objective functions has led to the development of several solution concepts, [8]. A Pareto-optimal solution is used when there is cooperation among the decision makers. For systems in which cooperation cannot be guaranteed, a Nash solution is employed [9]. Some systems have a structure in which one user is able to enforce his strategy upon another user. A solution concept for this type of system is known as a leader-follower or Stackelberg solution.

The Nash decision strategy arises frequently in systems with multiple decision makers. An inherent property of the Nash strategy is that it prevents decision makers from cheating. Any unilateral deviation by a decision maker from the Nash equilibrium incurs a greater cost for that decision maker. It is clear that the Nash strategy is a rational strategy for systems whose users do not cooperate.

The Nash solution concept arises often in economic contexts. Consider firms competing against each other in a market. Each firm seeks a production level for optimizing its cost function: profit. The firms do not cooperate in determining production levels. A Nash strategy may also be required for many estimation and control problems. In an estimation and control scheme there may be one performance index, e.g., minimum mean square, for estimating the parameters and a different index, e.g., quadratic, for controlling the system. The goals of these performance indices may oppose each other, and therefore, a Nash solution is required. A Nash game can even arise in a leader-follower setting. Consider a hierarchical structure in which there are several followers at the same level in the structure. The leader imposes his strategy but the followers are permitted to compete with each other. In this case, the followers are involved in both leader-follower and Nash games.

1.2. Determining Nash Strategies Under Uncertainties

When the system and cost functions are known to the decision makers, a Nash solution can be found. An explicit closed-form expression for a

Nash equilibrium exists for linear systems with quadratic performance indices
[10]. A decision maker having information about the plant and the others'
objectives can determine both his and the other players' Nash strategies.
However, when either the plant is not known or the cost functions are not
known to each decision maker, a player cannot determine a priori the Nash
equilibrium. This work investigates how a decision maker can use reaction
relations of the other decision makers for determining Nash equilibrium.

1.3. Organization of Thesis

In Section 2 a linear quadratic game is posed and an equilibrium
is proposed. In Section 3 it is shown that the proposed equilibrium is
equivalent to a Nash equilibrium. It is proven in Section 4 that algorithms
which are updated based upon the error in the estimated state cannot converge
to a value different than the Nash equilibrium. In Section 5 an algorithm
using reaction relations of the other decision makers is described. Finally,
an example using the algorithm is given in Section 6.

2. A PROPOSED EQUILIBRIUM FOR A LINEAR QUADRATIC GAME

In this section a linear quadratic game is introduced and its certainty-equivalent optimal inputs are determined. An equilibrium for the game is proposed and the defined equilibrium is explicitly calculated.

2.1. Formulation of the Linear Quadratic Game

Consider a linear time-invariant, discrete system described by

$$X_{k+1} = AX_k + B_1 U_{1_k} + B_2 U_{2_k} + W_k \qquad (2.1)$$

where X_k is the n-dimensional state vector at time k, and U_{1_k} and U_{2_k} are m_1 and m_2 dimensional input vectors to be chosen by Decision-Maker 1 (DM1) and Decision-Maker 2 (DM2) at time k, respectively. Assume that (A,B_1) and (A,B_2) are controllable. Also assume that W_k is an n-dimensional Gaussian random vector with $E\{W_k\} = 0$ and $E\{W_k W_k'\} = P$, the (n·n) covariance matrix. The single-stage cost function associated with DMi (i = 1,2) at time k is

$$J_{i_k} = E\{(X_{k+1} - C_i)' Q_i (X_{k+1} - C_i) + U_{i_k}' R_i U_{i_k}\} \qquad (2.2)$$

where R_i is an $(m_i \times m_i)$ positive definite matrix, Q_i is an (n×n) positive semi-definite matrix, and C_i is an n-dimensional vector. The state X_k is available to each DM at time k. The plant (2.1) is known to each DM. Each DM knows his cost function parameters, Q, R, and C, but he does not know the other DM's. It is assumed that each DM plays rationally, that is, he chooses his input U to minimize his cost (2.2).

As stated earlier, each DM is attempting to tune his control law to the reaction relations of the other DM. Since cooperation between the DM's cannot be enforced, it is desirable that each DM tune his control to reach a Nash equilibrium. The Nash solution to the N-stage linear quadratic game is given in [10]; however, this solution requires the DM's to know each other's cost function. It is possible that estimates of the cost parameters could be used in a dynamic programming solution to the N-stage problem, but the implementation of the estimation schemes may involve calculating conditional probability distributions, which can be difficult. On the contrary, the calculations involved in minimizing the single stage cost function are quite simple. If the DM's play the single stage game over and over while updating their control laws appropriately at each stage, and their control laws converge to the Nash solution, then the goals of the DM's have been met with relatively simple calculations.

2.2. The Certainty Equilvalent Control

Due to the quadratic nature of the cost functions, linear controls are assumed:

$$U_{1_k} = F_{1_k} X_k + G_{1_k} \tag{2.3}$$

$$U_{2_k} = F_{2_k} X_k + G_{2_k} \tag{2.4}$$

where F_{i_k} $(i = 1,2)$ is an $(m_i \times n)$ matrix and G_{i_k} is an m_i-dimensional vector. Each DM estimates the other's input and then formulates his own input so as to minimize his cost based on his estimate. The principle of certainty

equivalence is invoked here: The DM's replace the noise by its mean and estimate each other's inputs and play optimally for those estimates. The game proceeds to the next stage with the DM's repeating the procedure above. When the DM's estimate each other's input correctly, we say they have reached an equilibrium. It will be shown that this equilibrium is a unique Nash equilibrium.

With assumption (2.4), DM1 views the system (2.1) as

$$\hat{X}_{1_{k+1}} = (A + B_2 \hat{F}_{2_k}) X_k + B_2 \hat{G}_{2_k} + B_1 U_{1_k} \tag{2.5}$$

where $\hat{X}_{1_{k+1}}$ is DM1's estimate of the next state based on his estimate of DM2's input

$$\hat{U}_{2_k} = \hat{F}_{2_k} X_k + \hat{G}_{2_k}. \tag{2.6}$$

The symbol '^' indicates an estimated value. DM1 sees his cost as

$$\hat{J}_{1_k} = [(A + B_2 \hat{F}_{2_k}) X_k + B_2 \hat{G}_{2_k} + B_1 U_{1_k} - C_1]' Q_1 [(A + B_2 \hat{F}_{2_k}) X_k + B_2 \hat{G}_{2_k}$$
$$+ B_1 U_{1_k} - C_1] + U_{1_k}' R_1 U_{1_k}. \tag{2.7}$$

Minimization of (2.7) with respect to U_{1_k} yields

$$U_{1_k} = -(B_1' Q_1 B_1 + R_1)^{-1} B_1' Q_1 [(A + B_2 \hat{F}_{2_k}) X_k + B_2 \hat{G}_{2_k} - C_1]. \tag{2.8}$$

The positive definiteness assumptions on Q_1 and R_1 guarantee the existence of the term involving the inverse in (2.8).

It is seen that DM1's input is a function of his estimate of DM2's input. DM1's input can be decomposed as follows

$$F_{1_k}(\hat{F}_{2_k}) = -(B_1'Q_1B_1 + R_1)^{-1}B_1'Q_1(A + B_2\hat{F}_{2_k}) \tag{2.9}$$

$$G_{1_k}(\hat{G}_{2_k}) = -(B_1'Q_1B_1 + R_1)^{-1}B_1'Q_1(B_2\hat{G}_{2_k} - C_1). \tag{2.10}$$

Note that DM2's cost parameters Q_2, R_2, and C_2, which are unknown to DM1, do not enter into (2.9) or (2.10). In forming his input, DM1 knows A, B_1, B_2, Q_1, R_1, and C_1. Once he has obtained his estimates of $\hat{F}_{2_k}$ and $\hat{G}_{2_k}$, his optimal input is easily obtained. Also note that when the input is decomposed, $F_{1_k}(\hat{F}_{1_k})$ does not depend on $G_{1_k}(\hat{G}_{2_k})$ and vice versa. Similar results for U_{2_k} are also obtained:

$$F_{2_k}(\hat{F}_{1_k}) = -(B_2'Q_2B_2 + R_2)^{-1}B_2'Q_2(A + B_1\hat{F}_{1_k}) \tag{2.11}$$

$$G_{2_k}(\hat{G}_{1_k}) = -(B_2'Q_2B_2 + R_2)^{-1}B_2'Q_2(B_1\hat{G}_{1_k} - C_2). \tag{2.12}$$

2.3. A Proposed Equilibrium

Let the equilibrium be defined as when each DM's estimate of the other's input is correct. At equilibrium we then have

$$
\begin{aligned}
F_{1_k}(\hat{F}_{2_k}) = \hat{F}_{1_k} = F_{1_e} \qquad & G_{1_k}(\hat{G}_{2_k}) = \hat{G}_{1_k} = G_{1_e} \\
F_{2_k}(\hat{F}_{1_k}) = \hat{F}_{2_k} = F_{2_e} \qquad & G_{2_k}(\hat{G}_{1_k}) = \hat{G}_{2_k} = G_{2_e}
\end{aligned}
\tag{2.13}
$$

where F_{i_e} and G_{i_e} ($i = 1,2$) denote the equilibrium solutions. Substituting (2.9)-(2.12) into (2.13) and denoting for ease of notation

$$\alpha_1 = B_1'Q_1B_1 + R_1$$

$$\alpha_2 = B_2'Q_2B_2 + R_2,$$

(2.14)

we obtain the equilibrium solutions:

$$F_{1_e} = (I - \alpha_1^{-1}B_1'Q_1B_2\alpha_2^{-1}B_2'Q_2B_1)^{-1}\alpha_1^{-1}B_1'Q_1(B_2\alpha_2^{-1}B_2'Q_2 - I)A \qquad (2.15)$$

$$F_{2_e} = (I - \alpha_2^{-1}B_2'Q_2B_1\alpha_1^{-1}B_1'Q_1B_2)^{-1}\alpha_2^{-1}B_2'Q_2(B_1\alpha_1^{-1}B_1'Q_1 - I)A \qquad (2.16)$$

$$G_{1_e} = (I - \alpha_1^{-1}B_1'Q_1B_2\alpha_2^{-1}B_2'Q_2B_1)^{-1}\alpha_1^{-1}B_1'Q_1(C_1 - B_2\alpha_2^{-1}B_2'Q_2C_2) \qquad (2.17)$$

$$G_{2_e} = (I - \alpha_2^{-1}B_2'Q_2B_1\alpha_1^{-1}B_1'Q_1B_2)^{-1}\alpha_2^{-1}B_2'Q_2(C_2 - B_1\alpha_1^{-1}B_1'Q_1C_1). \qquad (2.18)$$

As stated above, α_1^{-1} and α_2^{-1} always exist. From the matrix inversion lemma [11], it is seen that the existence of

$$(I - \alpha_1^{-1}B_1'Q_1B_2\alpha_2^{-1}B_2'Q_2B_1)^{-1} \qquad (2.19)$$

implies the existence of

$$(I - \alpha_2^{-1}B_2'Q_2B_1\alpha_1^{-1}B_1'Q_1B_2)^{-1}. \qquad (2.20)$$

Hence, the existence of the equilibrium defined by (2.13) hinges on the existence of (2.19). Note that the equilibrium cannot be calculated by either DM a priori since it involves the other DM's cost parameters. If the inverse (2.19) exists, we define the equilibrium plant as the system (2.1) with the equilibrium inputs applied.

The equilibrium plant is

$$X_{k+1} = \{I + \mu_1(\xi_2 - I) + \mu_2(\xi_1 - I)\}AX_k + \mu_1[C_1 - \xi_2C_2] + \mu_2[C_2 - \xi_1C_1] \qquad (2.21)$$

where

$$\nu_i = B_i \gamma_i^{-1} \alpha_i^{-1} B_i' Q_i \qquad (2.22)$$

$$\xi_i = B_i \alpha_i^{-1} B_i' Q_i \qquad (2.23)$$

$$\gamma_1 = I - \alpha_1^{-1} B_1' Q_1 B_2 \alpha_2^{-1} B_2' Q_2 B_1 \qquad (2.24)$$

$$\gamma_2 = I - \alpha_2^{-2} B_2' Q_2 B_1 \alpha_1^{-1} B_1' Q_1 B_2. \qquad (2.25)$$

The steady state of this equilibrium plant is denoted X_{ss} and given by

$$X_{ss} = [I - \{I + \mu_1(\xi_2 - I) + \mu_2(\xi_1 - I)\}A]^{-1}[\mu_1(C_1 - \xi_2 C_2) + \mu_2(C_2 - \xi_1 C_1)]. \qquad (2.26)$$

The steady state exists if the eigenvalues of

$$\{I + \mu_1(\xi_2 - I) + \mu_2(\xi_1 - I)\}A \qquad (2.27)$$

are within the unit circle.

For the scalar system

$$x_{k+1} = a x_k + b_1 u_{1_k} + b_2 u_{2_k} + w_k \qquad (2.28)$$

and cost function

$$J_{i_k} = q_i(x_i - c_i)^2 + r_i u_i^2,$$

we obtain from (2.15)-(2.18)

$$f_{1_e} = -a q_1 b_1 r_1 / \triangle$$

$$f_{2_e} = -a q_2 b_2 r_2 / \triangle$$

$$g_{1_e} = -q_1 b_1 (q_2 b_2^2 (c_2 - c_1) - c_1 r_2)/\Delta$$

$$g_{2_e} = -q_2 b_2 (q_1 b_1^2 (c_1 - c_2) - c_2 r_1)/\Delta$$

where $\Delta = r_1 q_2 b_2^2 + r_2 q_1 b_1^2 + r_1 r_2$. Since Δ is never zero, the equilibrium always exists. The equilibrium plant is

$$x_{k+1} = (a r_1 r_2 x_k + q_1 b_1^2 c_1 r_2 + q_2 b_2^2 c_2 r_1)/\Delta.$$

The equilibrium plant is stable if $|a r_1 r_2/\Delta| < 1$. Equivalently, it is stable if

$$|a| < \frac{q_1 b_1^2}{r_1} + \frac{q_2 b_2^2}{r_2} + 1. \tag{2.29}$$

If the equilibrium scalar plant is stable, the steady state is

$$x_{ss} = \frac{q_1 b_1^2 r_2 c_1 + q_2 b_2^2 r_1 c_2}{r_1 b_2^2 q_2 + r_2 b_1^2 q_1 + r_1 r_2 - a r_1 r_2}. \tag{2.30}$$

The equilibrium steady state controls, $u_i = f_{i_e} x_{ss} + g_{i_e}$, are

$$u_{1_{ss}} = \frac{b_1 q_1 (b_2^2 q_2 (c_1 - c_2) + r_2 c_1 (1-a))}{b_1^2 q_1 r_2 + b_2^2 q_2 r_1 + (1-a) r_1 r_2} \tag{2.31}$$

$$u_{2_{ss}} = \frac{b_2 q_2 (b_1^2 q_1 (c_2 - c_1) + r_1 c_2 (1-a)}{b_1^2 q_1 r_2 + b_2^2 q_2 r_1 + (1-a) r_1 r_2}. \tag{2.32}$$

Although the existence of the equilibrium plant is guaranteed for the scalar case, the stability of the equilibrium system is not known a priori to either DM unless $|a| < 1$. If $|a| < 1$ then (2.29) is trivially satisfied. We note that for r_1 or r_2 sufficiently small, or q_1 or q_2

sufficiently large, any initially unstable plant, that is, $|a| > 1$, will have
a stable equilibrium system. A problem posed by this fact is how the DM's
would realize the equilibrium does not yield a stable equilibrium system
and how the DM's should readjust their r's and q's in order to create a
stable equilibrium.

3. THE NASH EQUILIBRIUM SOLUTION

In this section it is shown that if the proposed equilibrium in
(2.15)-(2.18) exists, then the equilibrium is a Nash equilibrium.

The input strategy $U^* = \{U_1^*, U_2^*, \ldots, U_m^*\}$ is defined to be a Nash
equilibrium solution if, for each $m \in M$, where M is the set of decision
makers, $J_m(U^*) \leq J_m(U^{m*}, U_m)$ where $U^{m*} \triangleq \{U_1^*, U_2^*, \ldots, U_{m-1}^*, U_{m+1}^*, \ldots, U_m^*\}$. In
our case with $M = 2$, $U^* = \{U_1^*, U_2^*\}$ is a Nash equilibrium if

$$J_1(U_1^*, U_2^*) \leq J_1(U_1, U_2^*) \qquad (3.1)$$

and

$$J_2(U_1^*, U_2^*) \leq J_2(U_1^*, U_2) \qquad (3.2)$$

for any U_1 and U_2. We prove that the equilibrium given by (2.15)-(2.18)
is Nash by verifying (3.1) and (3.2).

Suppose that U_{1_e} and U_{2_e} is a Nash equilibrium. Then we must have

$$J_1(U_{1_e}, U_{2_e}) \leq J_1(U_{1_e} + \delta_1, U_{2_e}) \qquad (3.3)$$

and

$$J_2(U_{1_e}, U_{2_e}) \leq J_1(U_{1_e}, U_{2_e} + \delta_2) \qquad (3.4)$$

for any arbitrary m_1-dimensional and m_2-dimensional vectors δ_1 and δ_2,
respectively. Let

$$\bar{U}_i = U_{i_e} + \delta_i = F_{i_e} X_k + G_{i_e} + \delta_i. \qquad (3.5)$$

Substituting (3.5) and (2.15)-(2.18) into (2.2) we obtain

$$J_i(\bar{U}_1, \bar{U}_2) = K_i + \delta_i'(B_i' Q_i B_i + R_i)\delta_i. \qquad (3.6)$$

See the Appendix for the expressions for K_i. Then

$$J_i(U_{1_e}, U_{2_e}) = J_i(\bar{U}_1, \bar{U}_2)\big|_{\delta_1 = \delta_2 = 0} = K_i$$

$$J_1(\bar{U}_1, U_{2_e}) = J_1(\bar{U}_1, \bar{U}_2)\big|_{\delta_2 = 0} = K_1 + \delta_1'(B_1'Q_1B_1 + R_1)\delta_1$$

$$J_2(U_{1_e}, \bar{U}_2) = J_2(\bar{U}_1, \bar{U}_2)\big|_{\delta_1 = 0} = K_2 + \delta_2'(B_2'Q_2B_2 + R_2)\delta_2.$$

Since $B_i'Q_iB_i + R_i$ is positive definite, the inequalities (3.3) and (3.4) are met.

The equilibrium in (2.15)-(2.18) is now known to be a Nash equilibrium. It follows that the equilibrium is unique provided the inverses in (2.15)-(2.18) exist. In solving for the equilibrium we assumed the inverses exist. If they do not exist, there are infinite solutions for (2.13). Since we desire an algorithm to converge to the Nash solution, we can only consider systems which yield a unique equilibrium. As in the scalar case, we can pose the problem of how the DM's should readjust their cost parameters to force a unique and stable equilibrium.

We have shown that if a solution exists for (2.13), then the solution is a unique Nash equilibrium. The solution to the finite horizon, N-stage, linear-quadratic Nash game when the plant and cost functions are known is given in [10]. For the case $N = 1$ and $C_i = 0$, we obtain a unique Nash equilibrium given by

$$F_{i_e} = -R_i^{-1}B_i'Q_i\Lambda^{-1}A, \tag{3.7}$$

if the inverse of $\Lambda = I + B_1R_1^{-1}B_1'Q_1 + B_2R_2^{-1}B_2'Q_2$ exists. We have shown that the defined equilibrium also exists given the existence of a specific matrix. We have attempted to verify algebraically that (3.7) is equivalent to (2.15) and (2.16). Various matrix identities were tried and the symbolic

processor REDUCE [12] was used. Even for the second-order system we were not able to prove equality for the general case. However, when numerical examples were examined, (2.15) and (2.16) yield the same results as (3.7). It is believed that with clever manipulation of (2.15) and (2.16) we could obtain (3.7).

At this point we justify estimating the other DM's F and G rather than his Q, R, and C. First, we estimate fewer parameters. In estimating F and G, DMi estimates $m_i \times (n+1)$ parameters; in estimating Q, R, and C, DMi estimates $n^2 + m_i^2 + m_i$ parameters. Second, in the expression for the equilibrium feedback $U_e = F_e X_k + G_e$, F_e and G_e are unique. The corresponding 3-tuple (Q,R,C) which generates the equilibrium feedback is, in general, not unique.

4. CONVERGENCE OF ALGORITHMS

We have seen that the equilibrium (2.13) leads to a unique Nash

equilibrium solution. In this section we examine the possibility of an

algorithm converging to a value different than the desired equilibrium.

It will be shown that it is not possible for the DM's to follow an equilibrium

trajectory while incorrectly estimating each other's control input. This

point is important for algorithms which update the estimates $\hat{F}$ and $\hat{G}$ based

on the error in the estimate of the next state.

4.1. The Error Equations

Consider the scalar case of (2.1)

$$x_{k+1} = ax_k + b_1 u_{1_k} + b_2 u_{2_k}. \tag{4.1}$$

In this development we assume the noise to be zero, because the probability

of the DM's estimating the next state correctly when the system is driven

by Gaussian noise is zero.

DM1 sees (4.1) as

$$\hat{x}_{1_{k+1}} = ax_k + b_1 u_{1_k} + b_2 \hat{u}_{2_k} \tag{4.2}$$

and DM2 sees the system as

$$\hat{x}_{2_{k+1}} = ax_k + b_1 \hat{u}_{1_k} + b_2 u_{2_k}. \tag{4.3}$$

At stage k the DM's apply their inputs u_k and estimate the next state $\hat{x}_{k+1}$.

At stage k+1, the DM's are given the state x_{k+1}. Each DM then formulates

his error which is the difference between the state x_{k+1} and the estimate $\hat{x}_{i_{k+1}}$:

$$e_{1_{k+1}} = x_{k+1} - \hat{x}_{1_{k+1}} \tag{4.4}$$

$$e_{2_{k+1}} = x_{k+1} - \hat{x}_{2_{k+1}}. \tag{4.5}$$

Substituting (4.1)-(4.3) into (4.4) and (4.5) we have

$$e_{1_{k+1}} = b_2(u_{2_k} - \hat{u}_{2_k}) \tag{4.6}$$

$$e_{2_{k+1}} = b_1(u_{1_k} - \hat{u}_{1_k}). \tag{4.7}$$

It is clear that, by definition, if the DM's play the equilibrium inputs (2.15)-(2.18), the errors (4.6) and (4.7) are zero. We intend to show that a necessary and sufficient condition for the errors in the estimates to be zero, and remain zero, is the inputs to (4.1) are the equilibrium inputs. We consider update laws of the form $\hat{f}_{i_{k+1}} = \hat{f}_{i_k} + \phi(e_{i_k})$ and $\hat{g}_{i_{k+1}} = \hat{g}_{i_k} + \theta(e_{i_k})$ where ϕ and θ are functions such that $\phi(0) = \theta(0) = 0$. This proposition is proved in the following two sections.

4.2. The Moving State Case

Clearly, if one of the errors is not zero, then the corresponding DM has incorrectly estimated the other's input. Suppose the errors are both zero and b_1 and b_2 are not zero. We have

$$e_{1_{k+1}} = u_{2_k} - \hat{u}_{2_k} = f_{2_k} x_k + g_{2_k} - (\hat{f}_{2_k} x_k + \hat{g}_{2_k}) = 0 \qquad (4.8)$$

$$e_{2_{k+1}} = u_{1_k} - \hat{u}_{1_k} = f_{1_k} x_k + g_{1_k} - (\hat{f}_{1_k} x_k + \hat{g}_{1_k}) = 0 \qquad (4.9)$$

which implies

$$f_{2_k} x_k + g_{2_k} = \hat{f}_{2_k} x_k + \hat{g}_{2_k} \qquad (4.10)$$

$$f_{1_k} x_k + g_{1_k} = \hat{f}_{1_k} x_k + \hat{g}_{1_k}. \qquad (4.11)$$

From (4.10) and (4.11), if $f_i = \hat{f}_i$ ($i = 1$ or 2) then $g_i = \hat{g}_i$, and if $g_i = \hat{g}_i$ then $f_i = \hat{f}_i$, provided $x_k \neq 0$. If this is true for both DM's then, by definition, the DM's are playing the equilibrium inputs. Suppose this is not true for $i = 2$. (A similar argument holds for $i = 1$.) Since the estimation errors (4.6) and (4.7) are both zero, the DM's do not update their estimates of $\hat{f}$ and $\hat{g}$ for the next input. This implies the inputs given by (2.9)-(2.12) remain the same. The errors of stage k+2 are

$$e_{1_{k+2}} = b_2(f_{2_{k+1}} x_{k+1} + g_{2_{k+1}} - (\hat{f}_{2_{k+1}} x_{k+1} + \hat{g}_{2_{k+1}})) = b_2(f_{2_k} x_{k+1} + g_{2_k}$$
$$- (\hat{f}_{2_k} x_{k+1} + \hat{g}_{2_1})) \qquad (4.12)$$

$$e_{2_{k+2}} = b_1(f_{1_{k+1}} x_{k+1} + g_{1_{k+1}} - (\hat{f}_{1_{k+1}} x_{k+1} + \hat{g}_{1_{k+1}})) = b_1(f_{1_k} x_{k+1} + g_{1_k}$$
$$- (\hat{f}_{1_k} x_{k+1} - \hat{g}_{1_k})). \qquad (4.13)$$

We now investigate whether the errors (4.12) and (4.13) can again be both zero if DM1 is not estimating DM2's control correctly. Let

$$y_{1_t} = f_{2_k} x_t + g_{2_k} \qquad (4.14)$$

$$y_{2_t} = \hat{f}_{2_k} x_t + \hat{g}_{2_k}, \qquad (4.15)$$

$t \in \{k, k+1\}$, and assume $f_{2_k} \neq \hat{f}_{2_k}$ and $g_{2_k} \neq \hat{g}_{2_k}$. A plot of the linear equations (4.14) and (4.15) given in Figure 1 illustrates the results. Clearly, if $x_{k+1} \neq x_k$ then $y_{1_{k+1}} \neq y_{2_{k+1}}$ and an error is generated. From the above we know that if there is an error in estimating the next state then the equilibrium inputs are not being applied. Although DM1 was able to estimate the state at k+1 correctly while not using the correct $\hat{f}_{2_k}$ and $\hat{g}_{2_k}$, this fault is revealed at stage k+2 if $x_{k+1} \neq x_k$. We see it is possible for both players to estimate the next state correctly even though one DM may not be playing the equilibrium input. However, if the state changes at the next stage, the error surfaces.

4.3. The Constant State Case

We now consider the case when the state remains constant, $x_{k+1} = x_k$. The errors (4.12) and (4.13) reduce to (4.8) and (4.9) which are zero. By simple mathematical induction we see that the state will remain constant for all future stages, the errors will remain zero, and therefore, there will be no updating of the controls. DM1 will continue to apply the wrong $\hat{f}_{2_k}$ and $\hat{g}_{2_k}$, but he will have no error in his prediction of the next state. Let us reconcile this difficulty by examining the properties of the constant, or steady state x_k. To be more general, let us not require DM2 to estimate $\hat{f}_{1_k}$ and $\hat{g}_{1_k}$ correctly either.
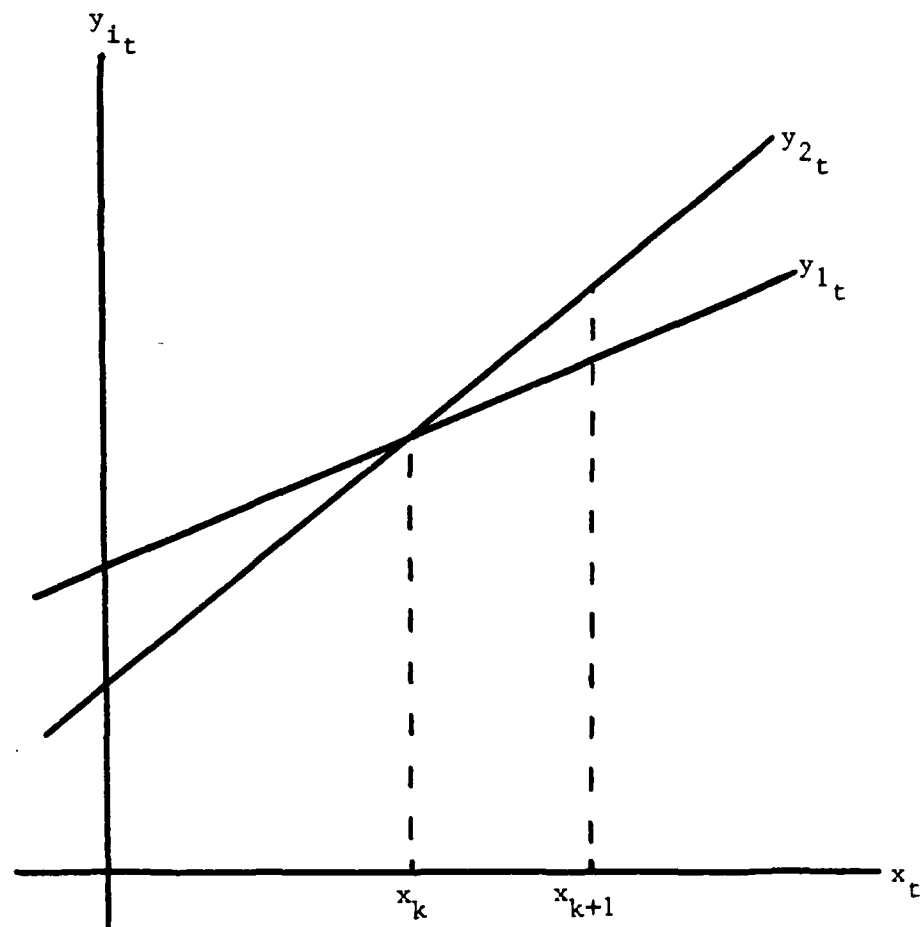
Figure 1. The linear equations (4.14) and (4.15) showing an error is generated if $x_{k+1} \neq x_k$.

We have

$$f_{2_k} x_k + g_{2_k} - (\hat{f}_{2_k} x_k + \hat{g}_{2_k}) = 0$$

$$f_{1_k} x_k + g_{1_k} - (\hat{f}_{1_k} x_k + \hat{g}_{1_k}) = 0$$

$$x_{k+1} = a x_k + b_1 (f_{1_k} x_k + g_{1_k}) + b_2 (f_{2_k} x_k + g_{2_k}) = x_k$$

with f_{1_k}, g_{1_k}, f_{2_k}, g_{2_k} given by (2.9)-(2.12), respectively, yielding

$$\left\{ a + b_1 \left(\frac{-q_1 b_1 (a + b_2 \hat{f}_{2_k})}{b_1^2 q_1 + r_1} \right) + b_2 \left(\frac{-q_2 b_2 (a + b_1 \hat{f}_{1_k})}{b_2^2 q_2 + r_2} \right) - 1 \right\} x_k + b_1 \left(\frac{-q_1 b_1 (b_2 \hat{g}_{2_k} - c_1)}{b_1^2 q_1 + r_1} \right)$$

$$+ b_2 \left(\frac{-q_2 b_2 (b_1 \hat{g}_{1_k} - c_2)}{b_2^2 q_2 + r_2} \right) = 0. \tag{4.16}$$

$$\frac{-q_2 b_2 (a + b_1 \hat{f}_{1_k})}{b_2^2 q_2 + r_2} x_k + \frac{-q_2 b_2 (b_1 \hat{g}_{1_k} - c_2)}{b_2^2 q_2 + r_2} - (\hat{f}_{2_k} x_k + \hat{g}_{2_k}) = 0 \tag{4.17}$$

$$\frac{-q_1 b_1 (a + b_2 \hat{f}_{2_k})}{b_1^2 q_1 + r_1} x_k + \frac{-q_1 b_1 (b_2 \hat{g}_{2_k} - c_1)}{b_1^2 q_1 + r_1} - (\hat{f}_{1_k} x_k + \hat{g}_{1_k}) = 0. \tag{4.18}$$

We have three equations in five variables: x_k, $\hat{f}_{1_k}$, $\hat{f}_{2_k}$, $\hat{g}_{1_k}$, $\hat{g}_{2_k}$. It appears we have two degrees of freedom and so let us fix x_k and $\hat{f}_{1_k}$ to any arbitrary values and solve the remaining system. (Similar results hold if we fix $\hat{f}_{2_k}$, $\hat{g}_{1_k}$, or $\hat{g}_{2_k}$.)

Solving (4.18) for $\hat{g}_{1_k}$ we obtain

$$\hat{g}_{1_k} = \frac{-1}{b_1^2 q_1 + r_1} [(q_1 b_1 (a + b_2 \hat{f}_{2_k}) + (b_1^2 q_1 + r_1) \hat{f}_{1_k}) x_k + q_1 b_1 (b_2 \hat{g}_{2_k} - c_1)]. \tag{4.19}$$

Substituting (4.19) into (4.17) and solving for $\hat{g}_{2_k}$ gives

$$\hat{g}_{2_k} = \frac{-1}{r_1 r_2 + r_1 b_2^2 q_2 + r_2 b_1^2 q_1} [W_1 x_k + W_2] \qquad (4.20)$$

where

$$W_1 = ab_2 q_2 (b_1^2 q_1 + r_1) - q_1 q_2 b_1^2 b_2 (a + b_2 \hat{f}_{2_k}) + (b_1^2 q_1 + r_1)(b_2^2 q_2 + r_2) \hat{f}_{2_k}$$

$$W_2 = q_1 q_2 b_1^2 b_2 c_1 - q_2 b_2 c_2 (b_1^2 q_1 + r_1).$$

Note that in (4.20) $\hat{g}_{2_k}$ has no dependence on $\hat{f}_{1_k}$. Finally, substituting (4.19) and (4.20) into (4.16) we obtain

$$x_k = \frac{q_1 b_1^2 r_2 c_1 + q_2 b_2^2 r_1 c_2}{r_1 q_2 b_2^2 + r_2 q_1 b_1^2 + r_1 r_2 - a r_1 r_2} . \qquad (4.21)$$

Note that the variable we were solving for, $\hat{f}_{2_k}$, drops out and does not appear in (4.21). In its place we have a requirement for the arbitrarily chosen x_k. If the state is not at (4.21), then the system of equations (4.16)-(4.18) is inconsistent. For a solution, or a set of solutions to exist, we must have the constant state at (4.21). We recognize this requirement as the steady state (2.30) of the equilibrium scalar plant. The DM's may estimate each other's control incorrectly with no error, and continue to have no error in estimating the next state only if the next state remains at the equilibrium steady state.

Substituting the steady state into the equations (4.16)-(4.18) and denoting the results as the steady state controls, we obtain

$$\hat{g}_{1_{ss}} = \frac{-(q_1 r_2 c_1 (\hat{f}_{1_k} b_1^2 c_1 + 1 - ab_1) + b_1 b_2^2 q_1 q_2 (c_1 - c_2) - \hat{f}_{1_k} b_2^2 q_2 r_1 c_2)}{b_1^2 q_1 r_2 + b_2^2 q_2 r_1 + r_1 r_2 - a r_1 r_2} \qquad (4.22)$$

$$\hat{g}_{2_{ss}} = \frac{-(q_2 r_1 c_2 (\hat{f}_{2_k} b_2^2 c_2 + 1 - ab_2) + b_1^2 b_2 q_1 q_2 (c_2 - c_1) - \hat{f}_{2_k} b_1^2 q_1 r_2 c_1)}{b_1^2 q_1 r_2 + b_2^2 q_2 r_1 + r_1 r_2 - ar_1 r_2} . \quad (4.23)$$

Note that $\hat{g}_{1_{ss}}$ is a function of $\hat{f}_{1_k}$ but not of $\hat{f}_{2_k}$, and $\hat{g}_{2_{ss}}$ is a function of $\hat{f}_{2_k}$ but not of $\hat{f}_{1_k}$. For the original system of equations to be consistent, this separation of estimation parameters must occur. We solve for each DM's estimate of the other's control, $\hat{u}_{i_{ss}} = \hat{f}_i x_{ss} + \hat{g}_{i_{ss}}$, from (4.21)-(4.23):

$$u_{1_{ss}} = \frac{b_1 q_1 (b_2^2 q_2 (c_1 - c_2) + r_2 c_1 (1-a))}{b_1^2 q_1 r_2 + b_2^2 q_2 r_1 + r_1 r_2 - ar_1 r_2} \quad (4.24)$$

$$u_{2_{ss}} = \frac{b_2 q_2 (b_1^2 q_1 (c_2 - c_1) + r_1 c_2 (1-a))}{b_1^2 q_1 r_2 + b_2^2 q_2 r_1 + r_1 r_2 - ar_1 r_2} . \quad (4.25)$$

We observe that even though $\hat{f}_{1_k}$ and $\hat{f}_{2_k}$ can be chosen arbitrarily, the resulting estimates of the steady state inputs are constants and equal to the equilibrium inputs (2.31) and (2.32). For the state to remain at the true equilibrium, the DM's must estimate that the other is playing the equilibrium input. This in turn causes the DM to play his own true equilibrium. Since the DM's are using the equilibrium inputs, they are not penalized for incorrectly estimating $\hat{f}$ and $\hat{g}$.

In summary, we see that if a DM estimates the next state incorrectly, then an error has been made in estimating the input parameters. If the DM's estimate the next state correctly and continue to estimate the next state correctly, then the inputs u have been estimated correctly. If the state is changing and the DM's estimate the inputs correctly, then they have also estimated the parameters $\hat{f}$ and $\hat{g}$ correctly. If the state is not changing

and the DM's estimate the inputs u correctly, then the state is at the steady state (2.30). At the steady state the dynamics of the system are lost. In this case, the DM's do not have to estimate the input parameters correctly when minimizing their costs.

The simple example given below demonstrates the steady state situation. Consider the system

$$x_{k+1} = ax_k + b_1 u_{1_k} + b_2 u_{2_k}$$

$$J_{1_k} = q_1 x_2^2 + r_1 u_{1_k}^2$$

$$J_{2_k} = q_2 x_k^2 + r_2 u_{2_k}^2$$

with the control laws

$$u_{1_k} = f_{1_k} x_k$$

$$u_{2_k} = f_{2_k} x_k.$$

An obvious solution for minimizing the cost functions is $x_k = 0$ and $u_{i_k} = 0$. At the steady state, $x_{ss} = 0$, each DM estimates the other's input will be zero and applies his input $u_i = 0$. The DM's do not have to estimate the equilibrium $\hat{f}_k$'s; for any $\hat{f}_k$ their corresponding input will be the correct value, $u = 0$.

4.4. The Vector Case

We now briefly examine the convergence of algorithms for the general vector problem. Following the development of Sections 4.1 and 4.2, we obtain

$$E_{1_{k+1}} = X_{k+1} - \hat{X}_{1_{k+1}} = B_2(U_{2_k} - \hat{U}_{2_k}) \tag{4.26}$$

$$E_{2_{k+1}} = X_{k+1} - \hat{X}_{2_{k+1}} = B_1(U_{1_k} - \hat{U}_{1_k}). \tag{4.27}$$

The errors can be zero if the difference $U_{i_k} - \hat{U}_{i_k}$ is zero or if the difference is in the nullspace of B_i. To eliminate the latter possibility, we now require B_1 and B_2 to have full column rank. This requirement is not too restrictive; it is equivalent to having no redundant control. With this requirement the errors are zero if

$$(F_{2_k} - \hat{F}_{2_k})X_k + (G_{2_k} - \hat{G}_{2_k}) = 0$$

and

$$(F_{1_k} - \hat{F}_{1_k})X_k + (G_{1_k} - \hat{G}_{1_k}) = 0.$$

We conjecture results for convergence, similar to those obtained for the scalar, case could be obtained. The matrix algebra required to justify our conjecture may be formidable. We realize there may be a possibility of converging to incorrect values if $G_{i_k} - \hat{G}_{i_k}$ lands in the range space of $F_{i_k} - \hat{F}_{i_k}$, but we believe this is unlikely.

5. A PROPOSED ALGORITHM

In this section an algorithm is proposed for determining the Nash equilibrium from the reaction relations of the decision makers. Several gradient-type algorithms are available for estimating unknown parameters [13],[14]. Convergence of these schemes has been shown for the single input case. We maintain the spirit of these algorithms and extend them to systems with multiple users.

We assume a decision maker can remember his previous L inputs, his previous L estimates of the other's inputs, and the previous L states. The number L is a finite memory buffer size whose value depends on the order of the system. For the noiseless case, DM1 can solve (4.26) for DM2's previous input U_{2_k}. Knowing X_k and U_{2_k} is not enough to determine F_{2_k} and G_{2_k}. However, if F_2 and G_2 are not changing rapidly, DM1 can consider them to be constant over the last L stages. DM1 can then use a least-squares scheme for determinng the best estimate of DM1's input parameters F_{2_k} and G_{2_k}. We denote these estimates as $\tilde{F}_{2_k}$ and $\tilde{G}_{2_k}$. Now DM1 can use the following updating scheme:

$$\hat{F}_{2_{k+1}} = \hat{F}_{2_k} + \frac{1}{2} (\tilde{F}_{2_k} - \hat{F}_{2_k}) \qquad (5.1)$$

$$\hat{G}_{2_{k+1}} = \hat{G}_{2_k} + \frac{1}{2} (\tilde{G}_{2_k} - \hat{G}_{2_k}). \qquad (5.2)$$

If $\tilde{F}_{2_k} = F_{2_k}$ then a simple interpretation of the updating scheme is: DM1's next choice of $\hat{F}_2$ is the average of his last estimate and DM2's actual last input. We note the scheme in (5.1) and (5.2) has the desirable property of not updating when there is no error in estimating the previous input.

The algorithm is initialized with the assumption that the decision makers have similar costs and objectives. DM1's first estimates $\hat{F}_{2_o}$ and $\hat{G}_{2_o}$ are found by replacing DM2's unknown cost parameters Q_2, R_2, and C_2 with DM1's parameters Q_1, R_1, and C_1. DM2 does likewise.

As noted above, convergence for the stochastic gradient algorithms has been proven for the single user case. We have not shown they converge for the multiple user case. However, from our results of Section 4, if the proposed algorithm converges, it must converge to the correct values.

6. A WORKING EXAMPLE

Consider the system

$$X_{k+1} = \begin{pmatrix} 2 & 5 \\ 7 & 3 \end{pmatrix} X_k + \begin{pmatrix} 5 & 9 \\ 1 & 3 \end{pmatrix} U_{1_k} + \begin{pmatrix} 2 & 6 \\ 9 & 8 \end{pmatrix} U_{2_k} + \sigma W_k, \quad X_o = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

with cost functions

$$J_1 = \left(X_k - \begin{pmatrix} 9 \\ 5 \end{pmatrix} \right)' \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \left(X_k - \begin{pmatrix} 9 \\ 5 \end{pmatrix} \right) + U_{1_k}' \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix} U_{1_k}$$

$$J_2 = \left(X_k - \begin{pmatrix} 8 \\ -3 \end{pmatrix} \right)' \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \left(X_k - \begin{pmatrix} 8 \\ -3 \end{pmatrix} \right) + U_{2_k}' \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} U_{1_k}.$$

W_k is a diagonal matrix whose entries are from a zero mean, unit variance, Gaussian distribution. The noise level is scaled with the factor σ.

The matrix $\begin{pmatrix} 2 & 5 \\ 7 & 3 \end{pmatrix}$ is not stable because it has an eigenvalue in the left half plane, and so control is required to stabilize the plant.

In this example the DM's know the other's Q and R but do not know the other's target vector C. The DM's can calculate a priori the equilibrium F's, but must estimate each other's G vector.

Solving (2.15)-(2.18) and (2.21) for the example above, we obtain

$$F_{1_e} = \begin{bmatrix} 0.055 & -0.158 \\ 0.061 & -0.274 \end{bmatrix} \qquad F_{2_e} = \begin{bmatrix} -0.518 & 0.019 \\ -0.308 & -0.278 \end{bmatrix}$$

$$G_{1_e} = \begin{bmatrix} 0.249 \\ 3.543 \end{bmatrix} \qquad G_{2_e} = \begin{bmatrix} 2.384 \\ -4.545 \end{bmatrix}$$

$$X_{ss} = \begin{bmatrix} 9.711 \\ -2.880 \end{bmatrix}.$$

The following figures show results of simulations with systems having no noise, $\sigma = 0$, and systems having noise, $\sigma = 0.1$. The figures display the actual state and its estimates $\hat{X}_1$ and $\hat{X}_2$, and the controls G_1 and G_2 and their estimates $\hat{G}_1$ and $\hat{G}_2$. Since the state and the control G_i are two-dimensional vectors, we display the components of these vectors one at a time. We use the following notation to indicate the components:

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \hat{X}_1 = \begin{bmatrix} \hat{x}_{11} \\ \hat{x}_{12} \end{bmatrix} \qquad \hat{X}_2 = \begin{bmatrix} \hat{x}_{21} \\ \hat{x}_{22} \end{bmatrix}$$

$$G_1 = \begin{bmatrix} g_{11} \\ g_{12} \end{bmatrix} \qquad G_2 = \begin{bmatrix} g_{21} \\ g_{22} \end{bmatrix}$$

$$\hat{G}_1 = \begin{bmatrix} \hat{g}_{11} \\ \hat{g}_{12} \end{bmatrix} \qquad \hat{G}_2 = \begin{bmatrix} \hat{g}_{21} \\ \hat{g}_{22} \end{bmatrix} .$$

The results of simulations with $\sigma = 0$ are given in Figures 2-7. In Figures 2 and 3 we see how the estimates $\hat{X}_{i_k}$ converge to the actual state X_k. We also note that the state converges to the true steady state value. Figures 4-7 show the estimates $\hat{G}_{i_k}$ converging to the inputs G_{i_k}. We also see that the inputs G_{i_k} converge to the equilibrium values G_{i_e}.

The results of simulations with $\sigma = 0.1$ are given in Figures 8-15. Figures 8-11 show the state and its estimates and Figures 12-15 show the control and its estimates. The effect of the noise is evident in the plots of the actual state. The estimates of the state do not fluctuate as much as the actual state, because the actual state is driven by the noise. We again note that the estimates go to their equilibrium values. We
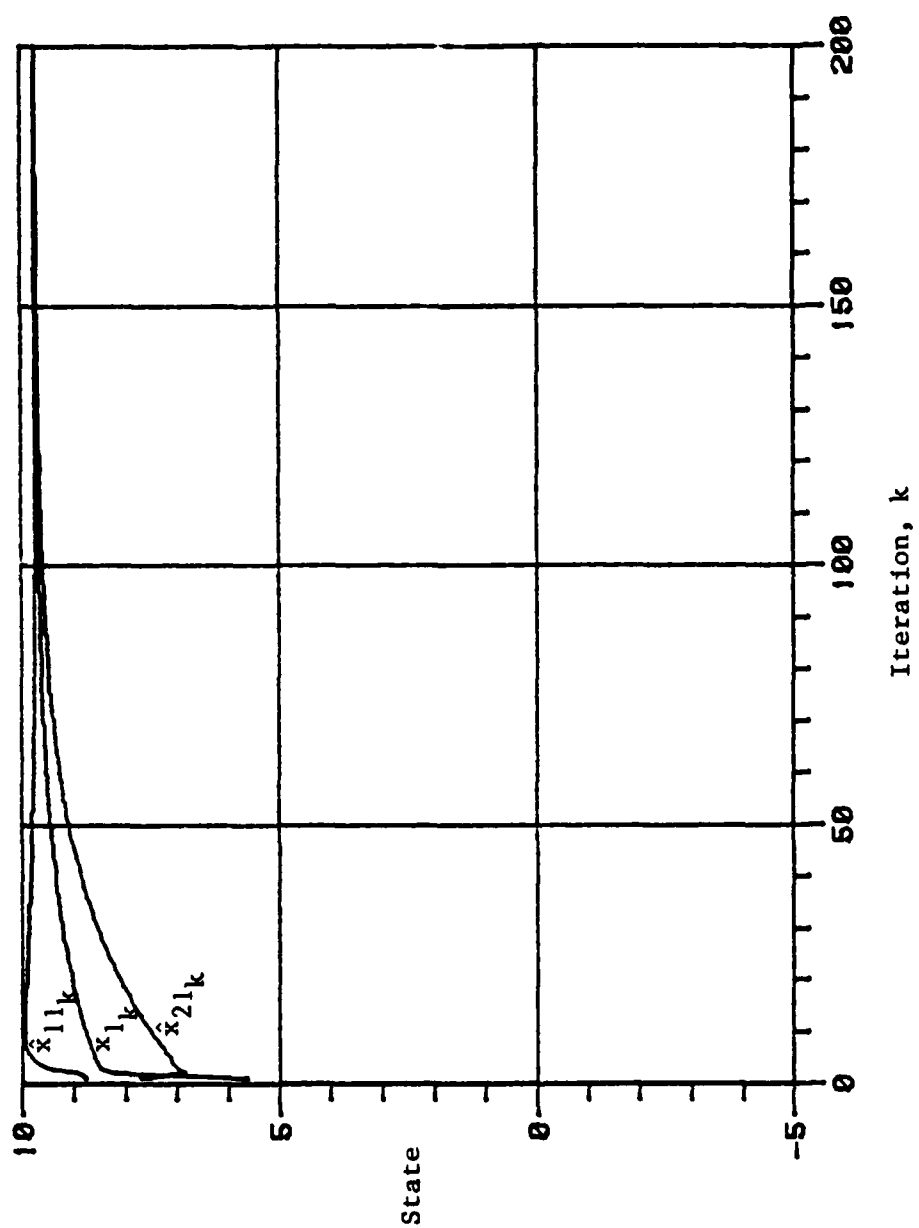
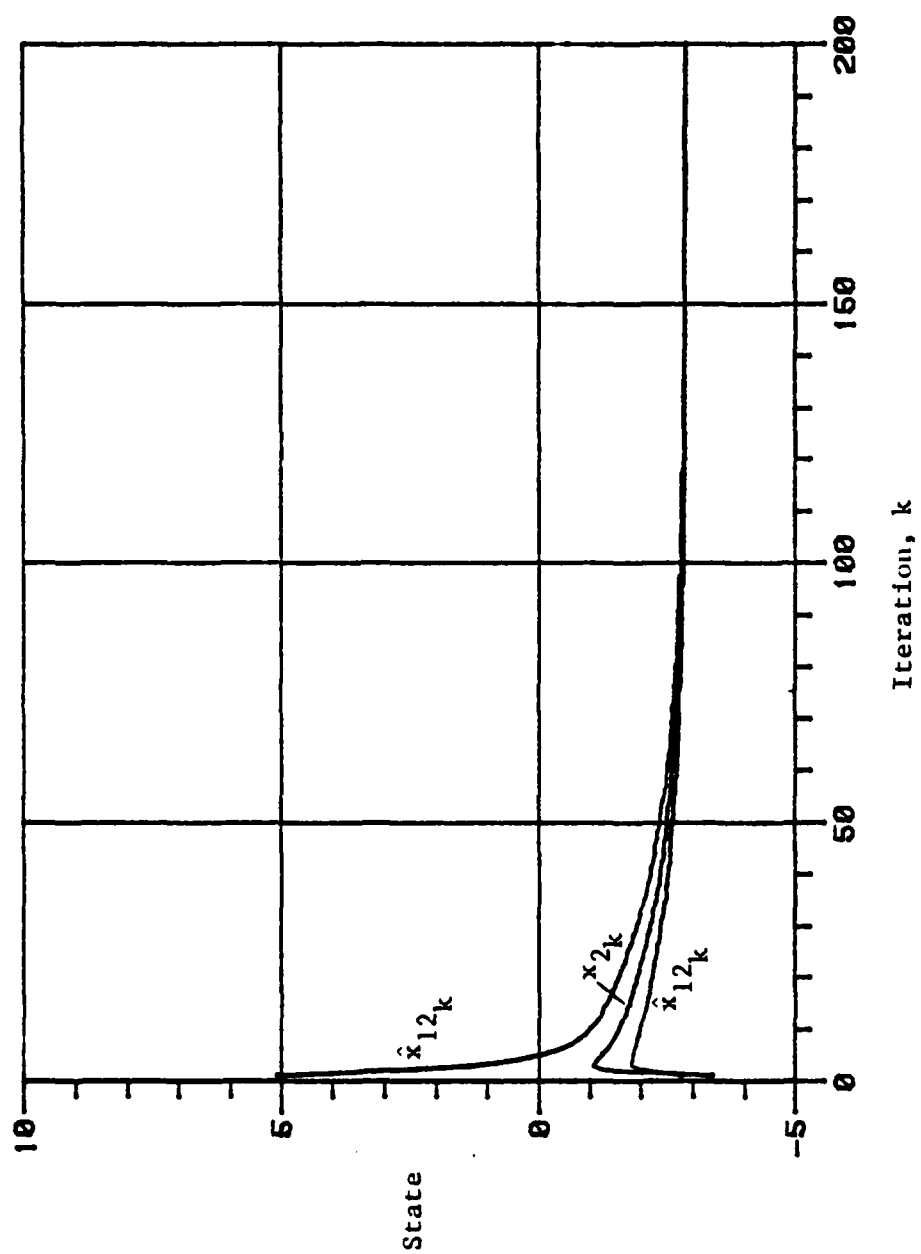Figure 2. The state x_{1_k} and its estimates $\hat{x}_{11_k}$ and $\hat{x}_{21}$ for $\sigma = 0$.

Figure 3. The state x_{2_k} and its estimates $\hat{x}_{12_k}$ and $\hat{x}_{22_k}$ for $\sigma = 0$.
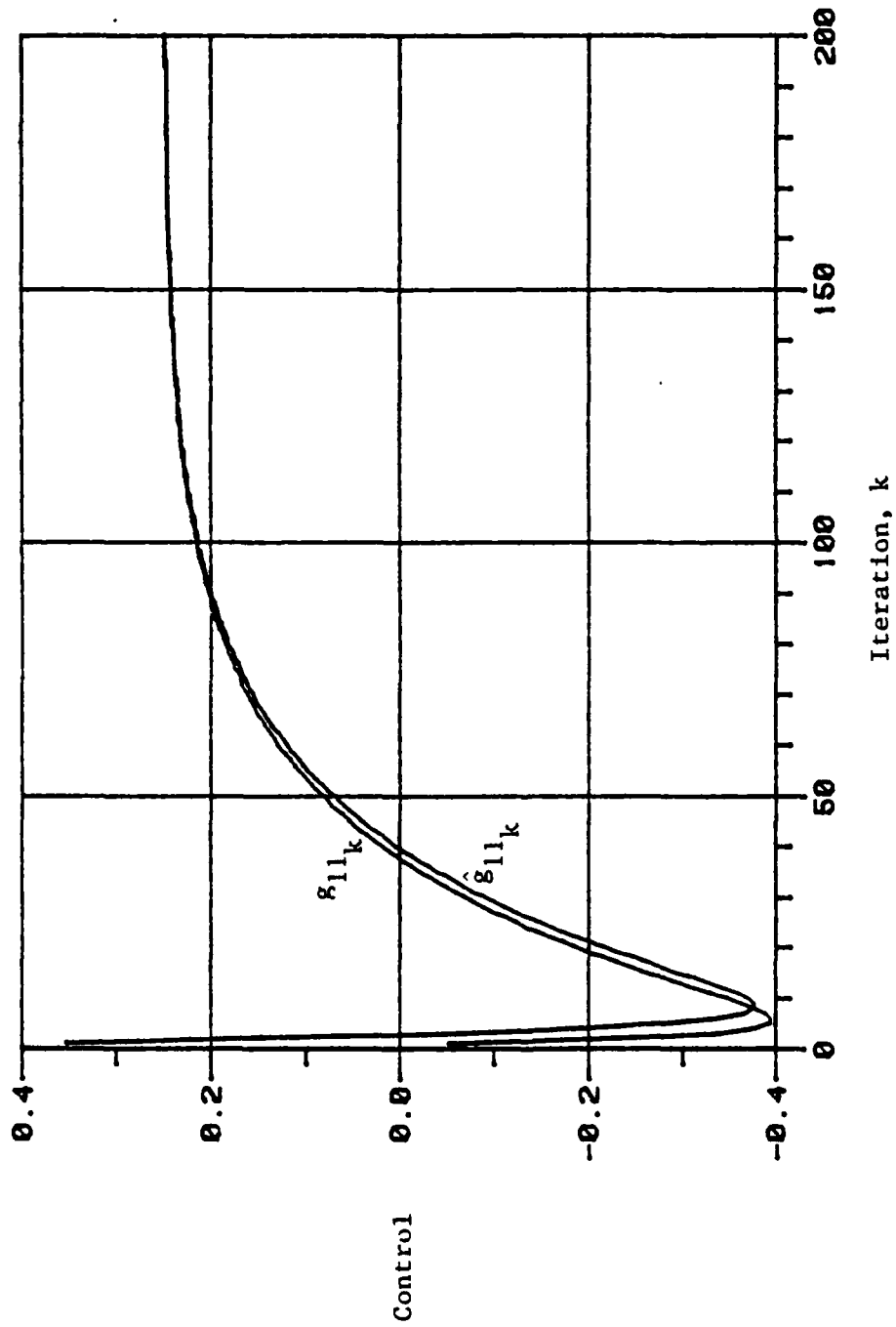
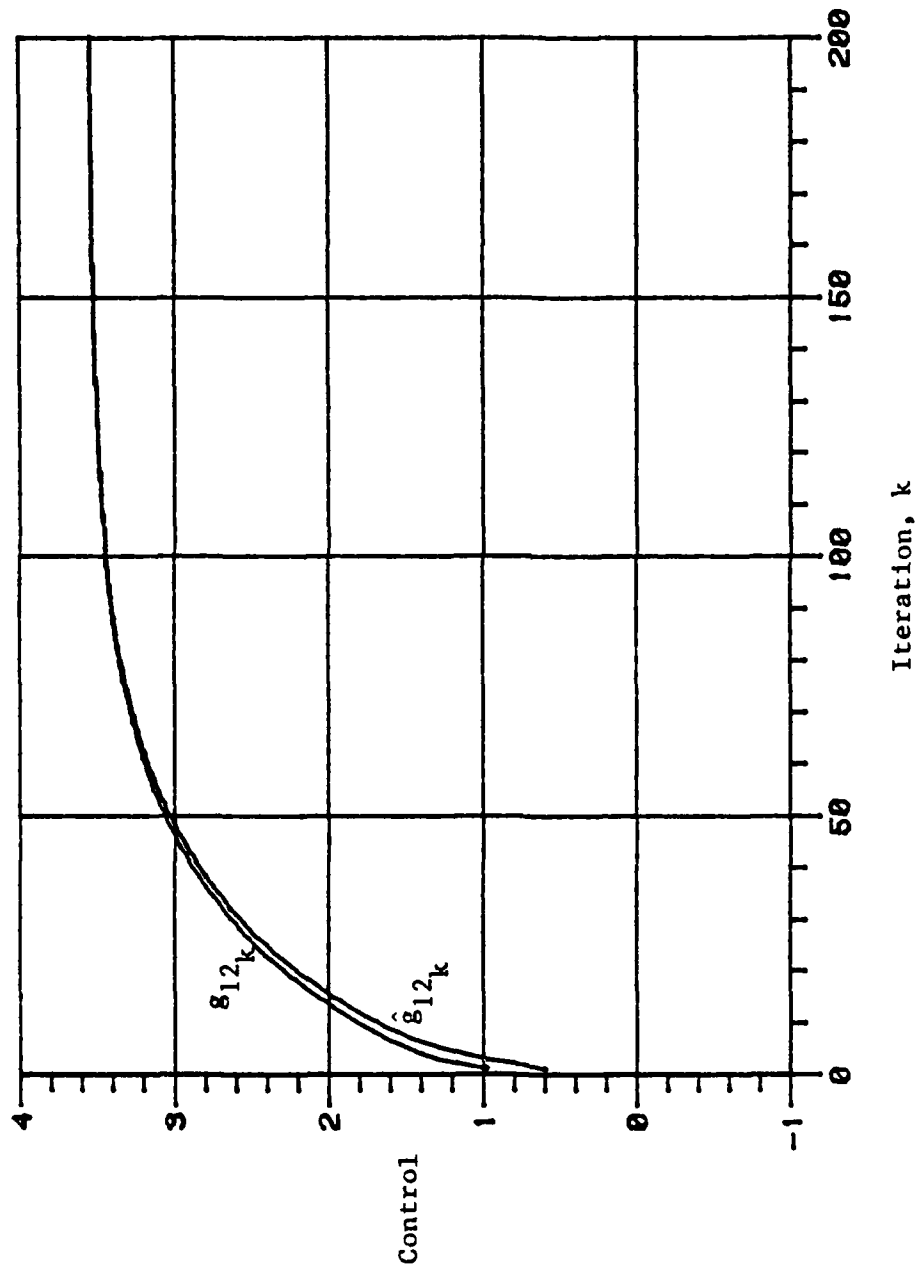Figure 4. The control g_{11_k} and its estimate $\hat{g}_{11_k}$ for $\sigma = 0$.

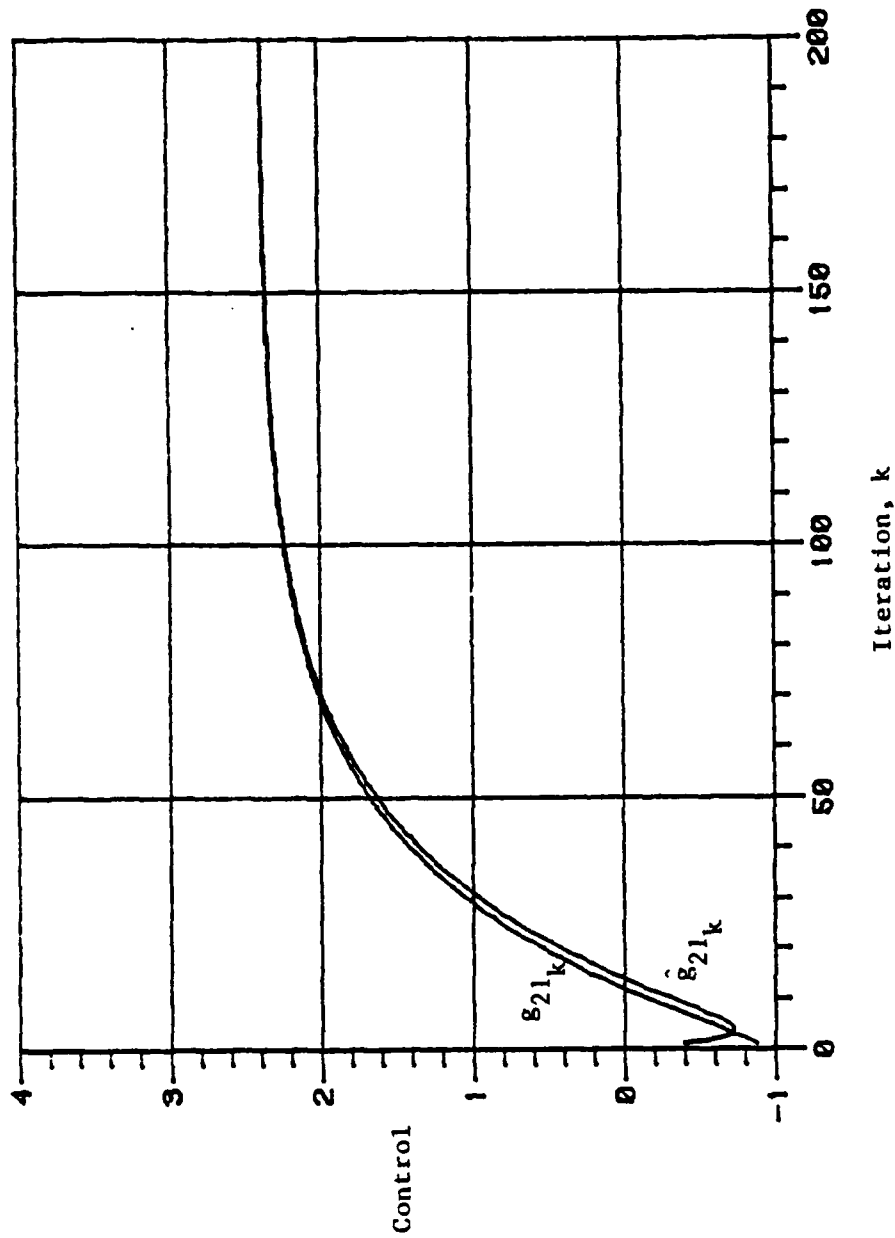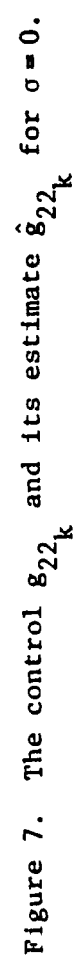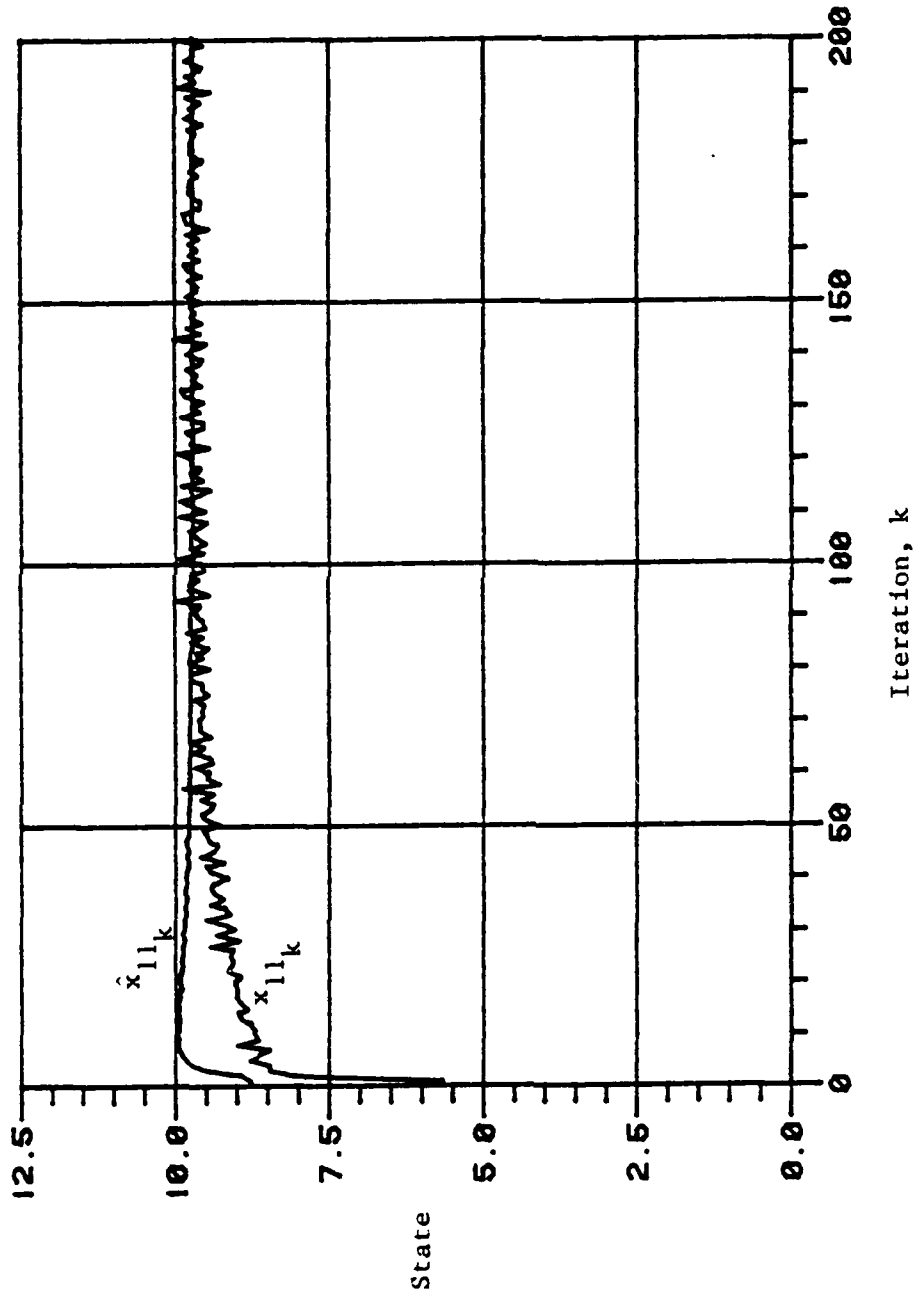Figure 5. The control g_{12_k} and its estimate $\hat{g}_{12_k}$ for $\sigma = 0$.

Figure 6. The control g_{21_k} and its estimate $\hat{g}_{21_k}$ for $\sigma = 0$.

Figure 7. The control g_{22_k} and its estimate $\hat{g}_{22_k}$ for $\sigma = 0$.

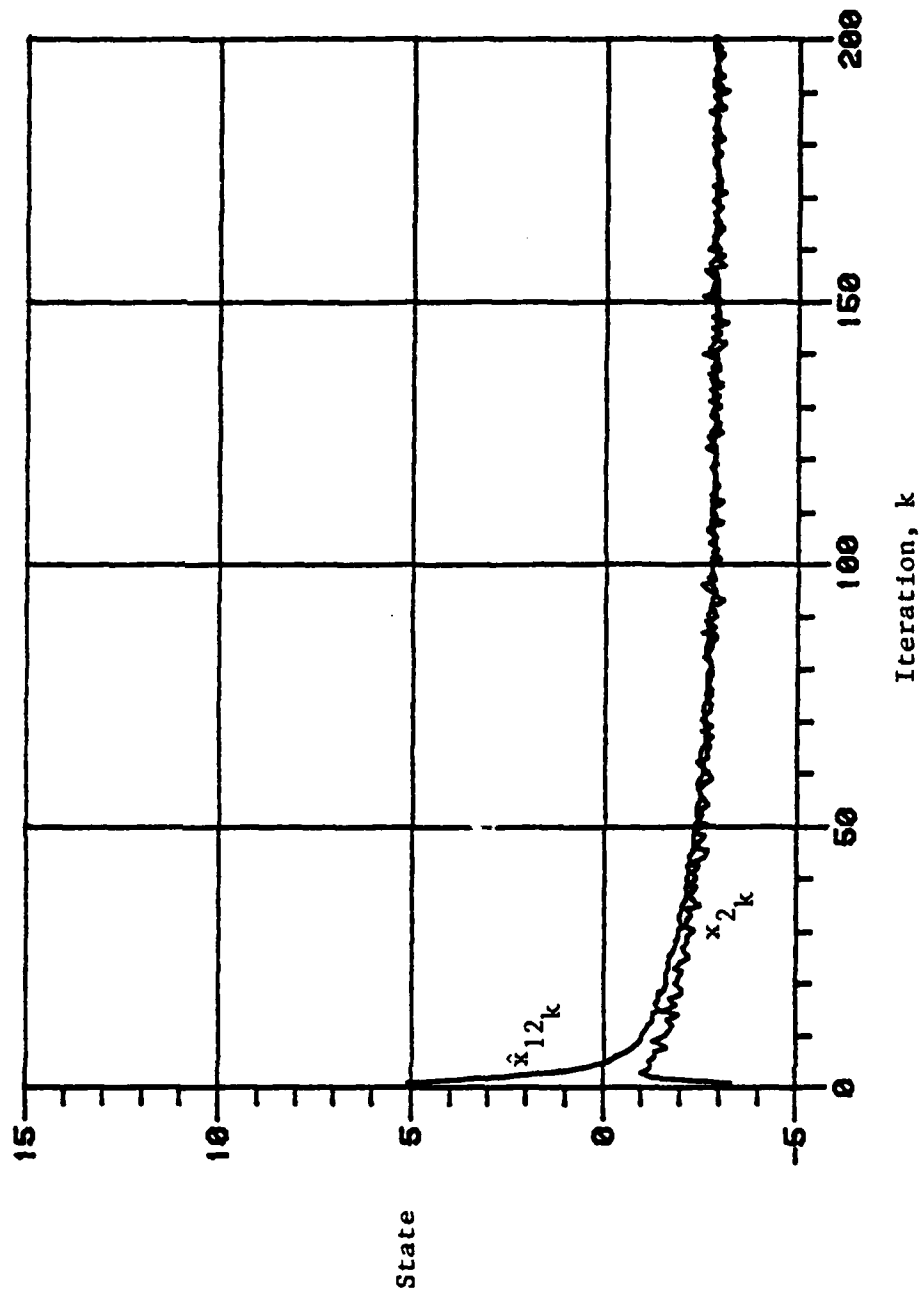Figure 8. The state x_{11_k} and its estimate $\hat{x}_{11_k}$ for $\sigma = 0.1$.
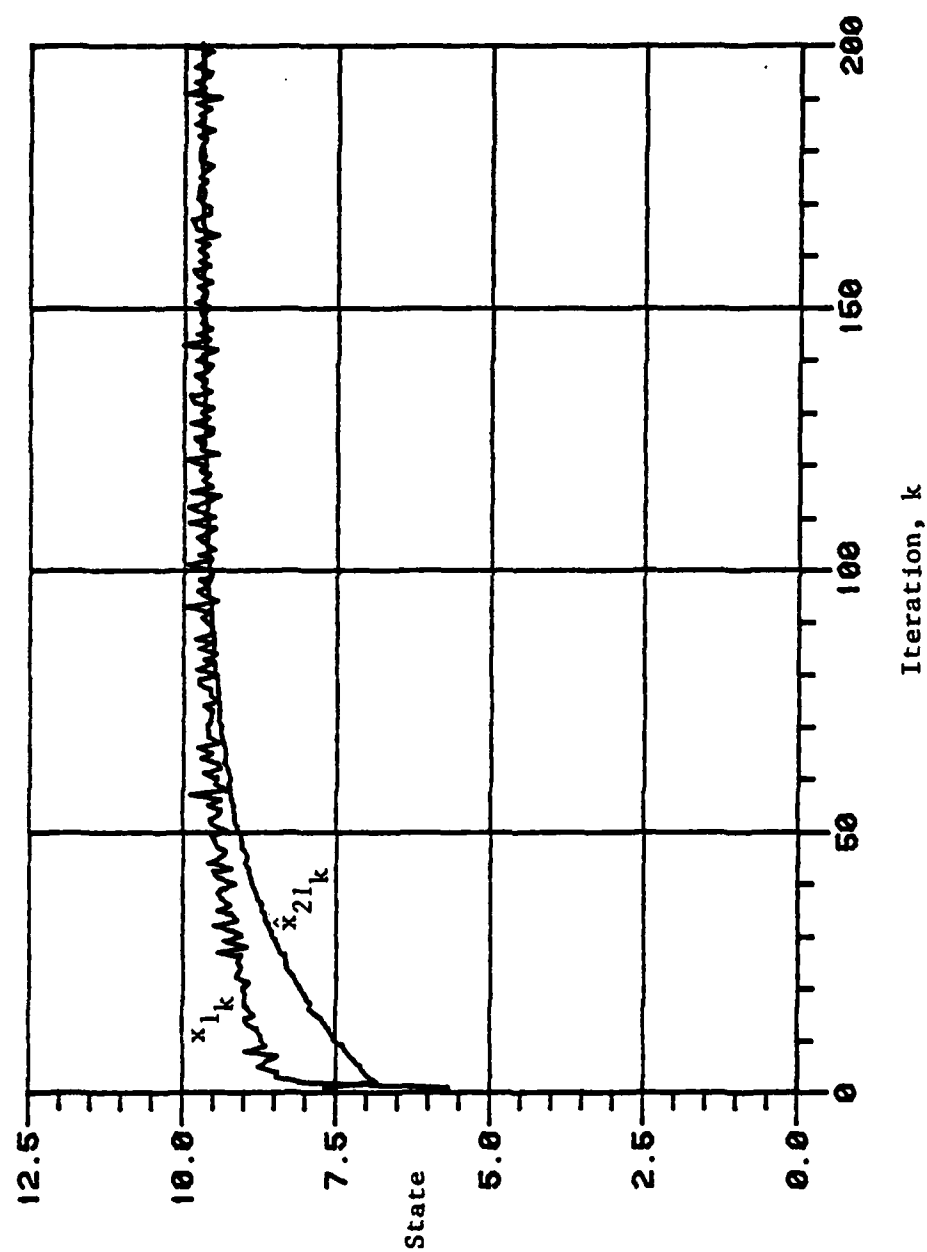
Figure 9. The state x_{1_k} and its estimate $\hat{x}_{21_k}$ for $\sigma = 0.1$.

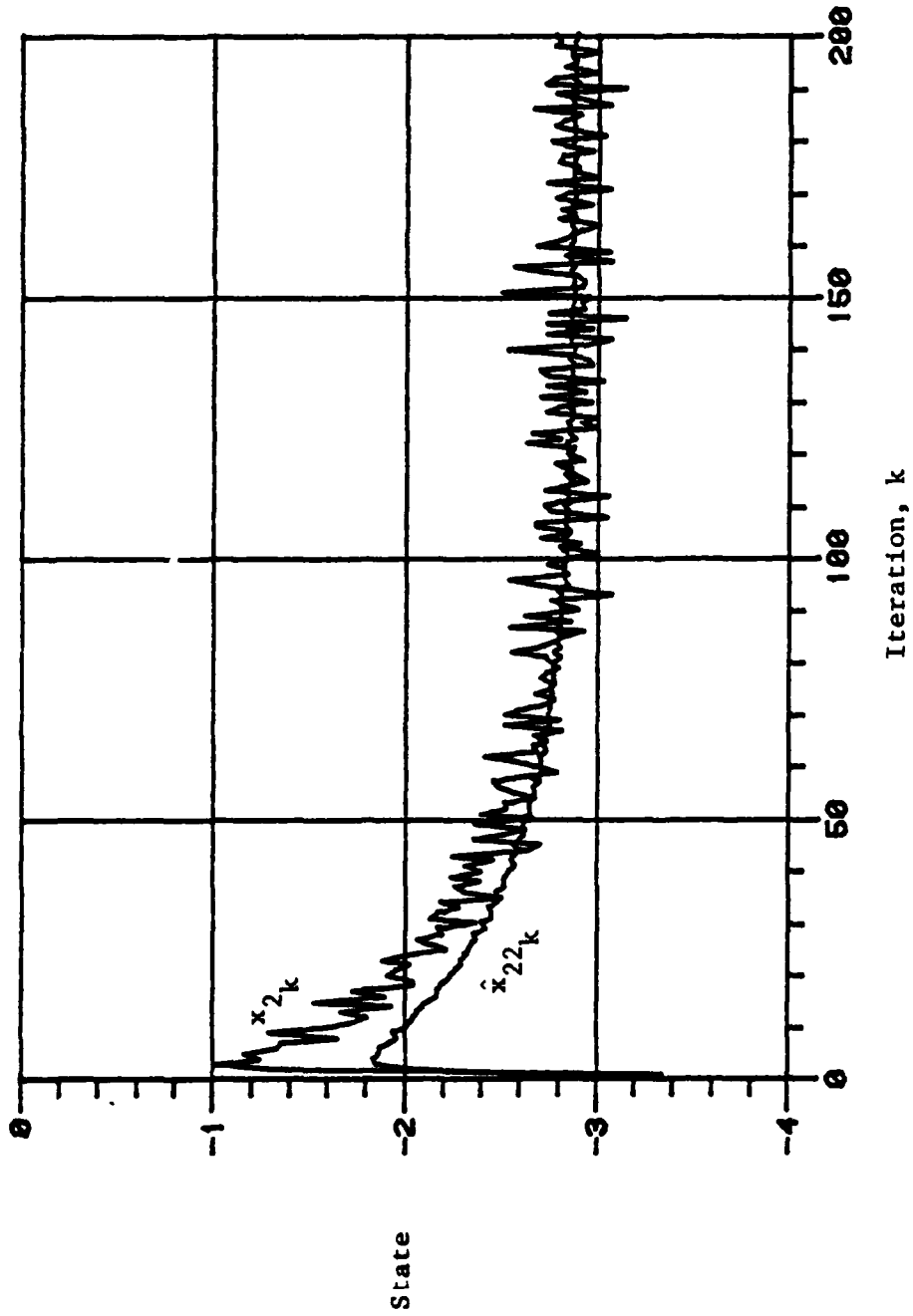Figure 10. The state x_{2_k} and its estimate $\hat{x}_{12_k}$ for $\sigma = 0.1$.

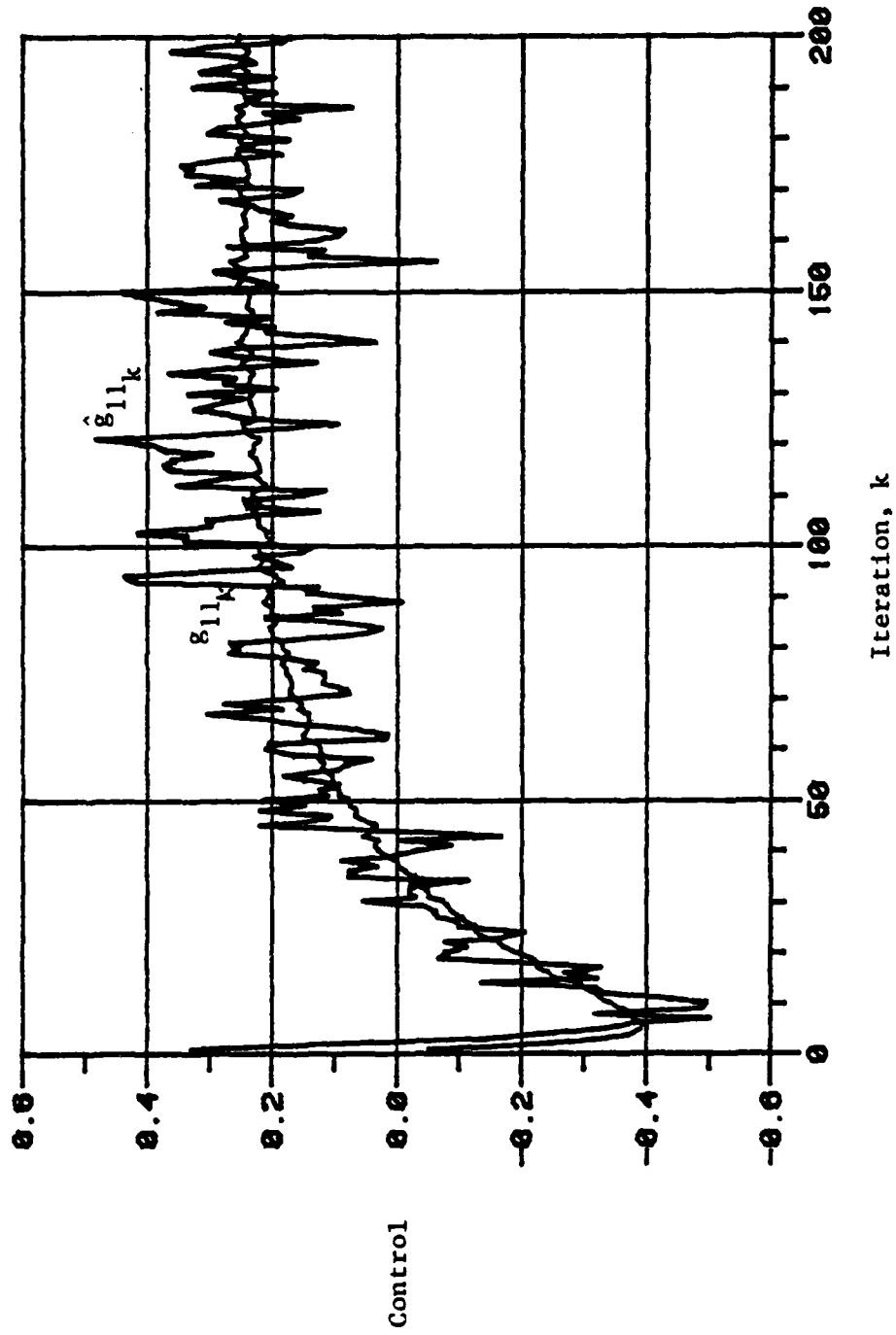Figure 11. The state x_{2_k} and its estimate $\hat{x}_{22_k}$ for $\sigma = 0.1$.

Figure 12. The control g_{11_k} and its estimate $\hat{g}_{11_k}$ for $\sigma = 0.1$.
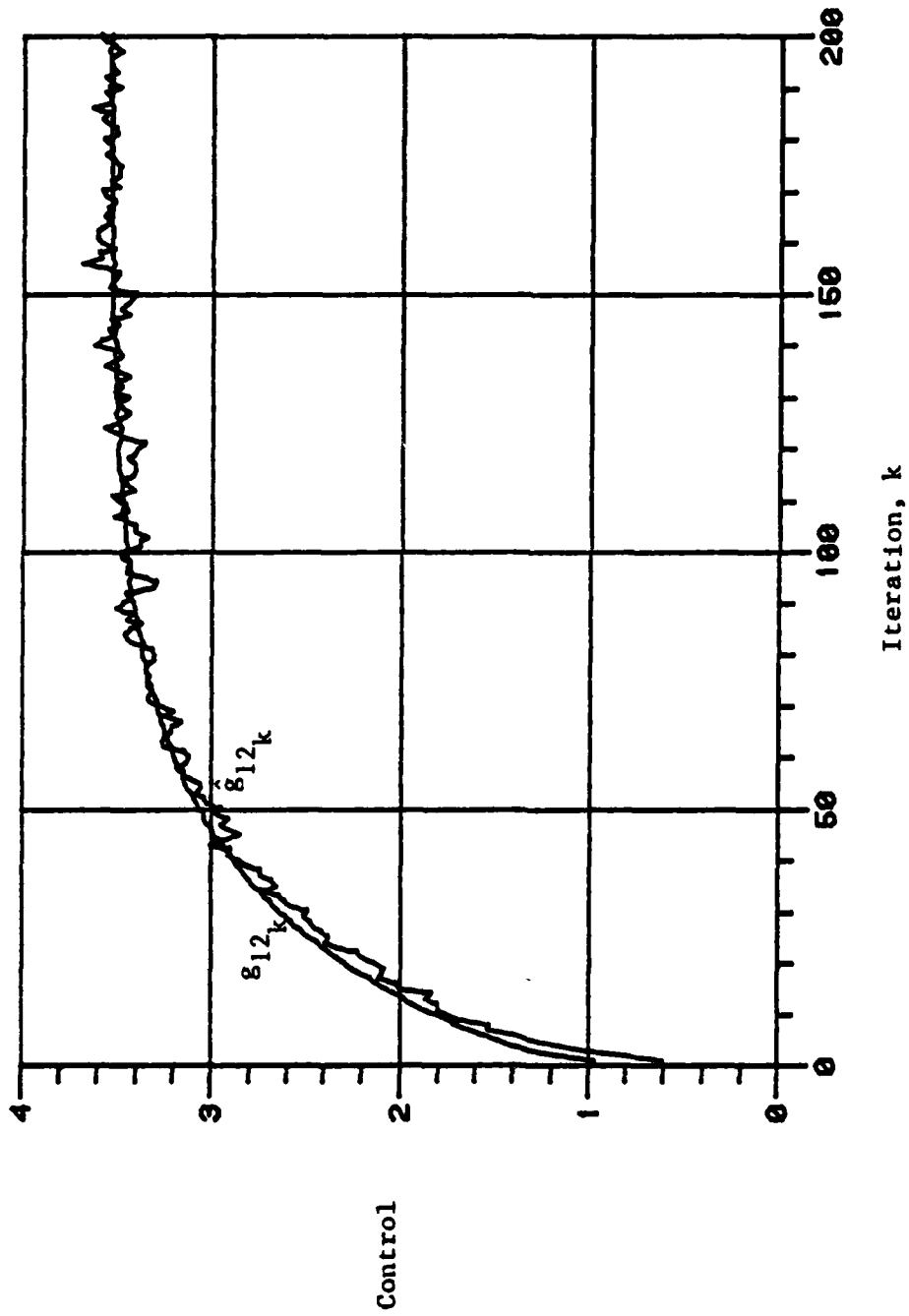
Figure 13. The control g_{12_k} and its estimate $\hat{g}_{12_k}$ for $\sigma = 0.1$.

Figure 14. The control g_{21_k} and its estimate $\hat{g}_{21_k}$ for $\sigma = 0.1$.

Figure 15. The control g_{22_k} and its estimate $\hat{g}_{22_k}$ for $\sigma = 0.1$.
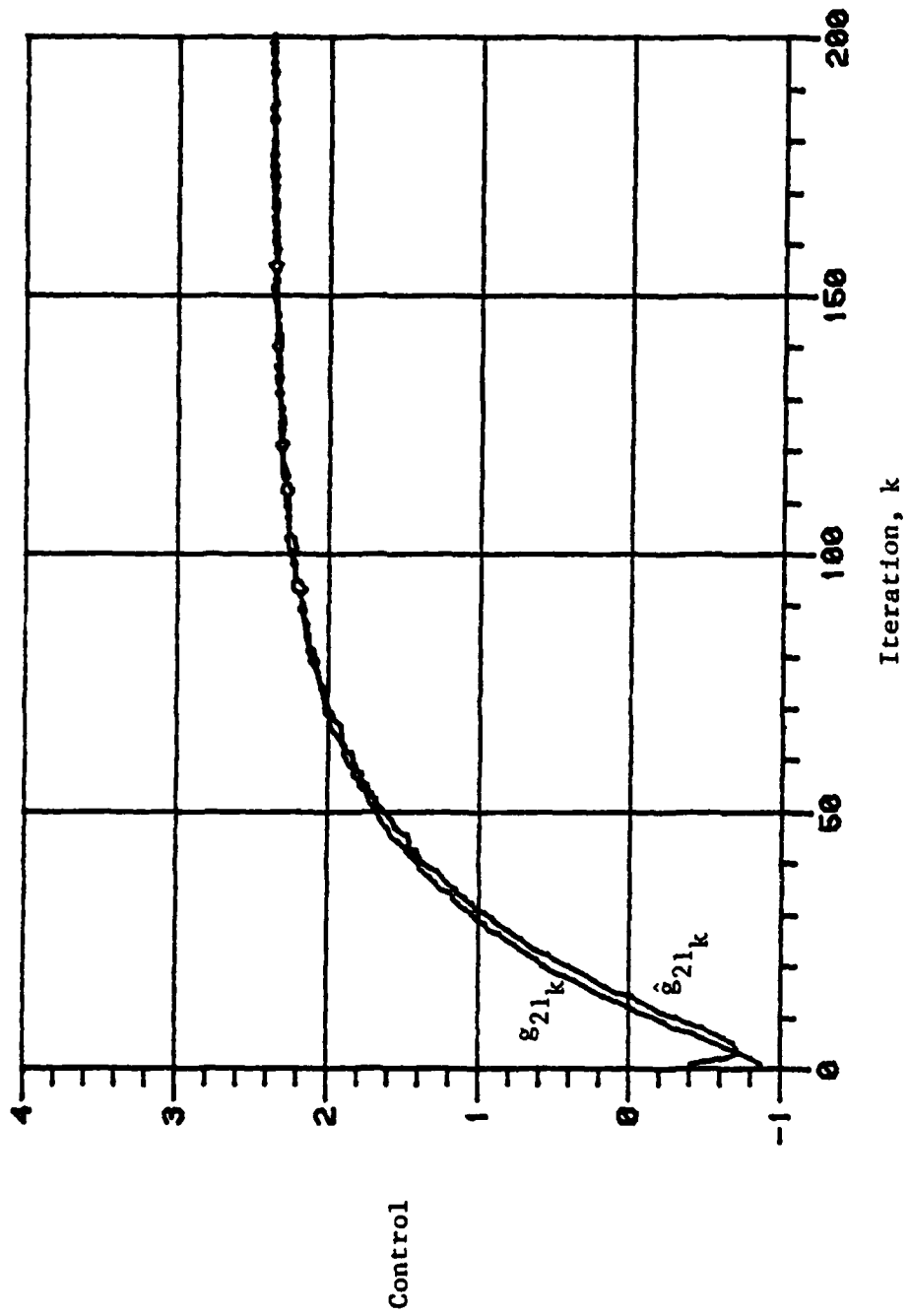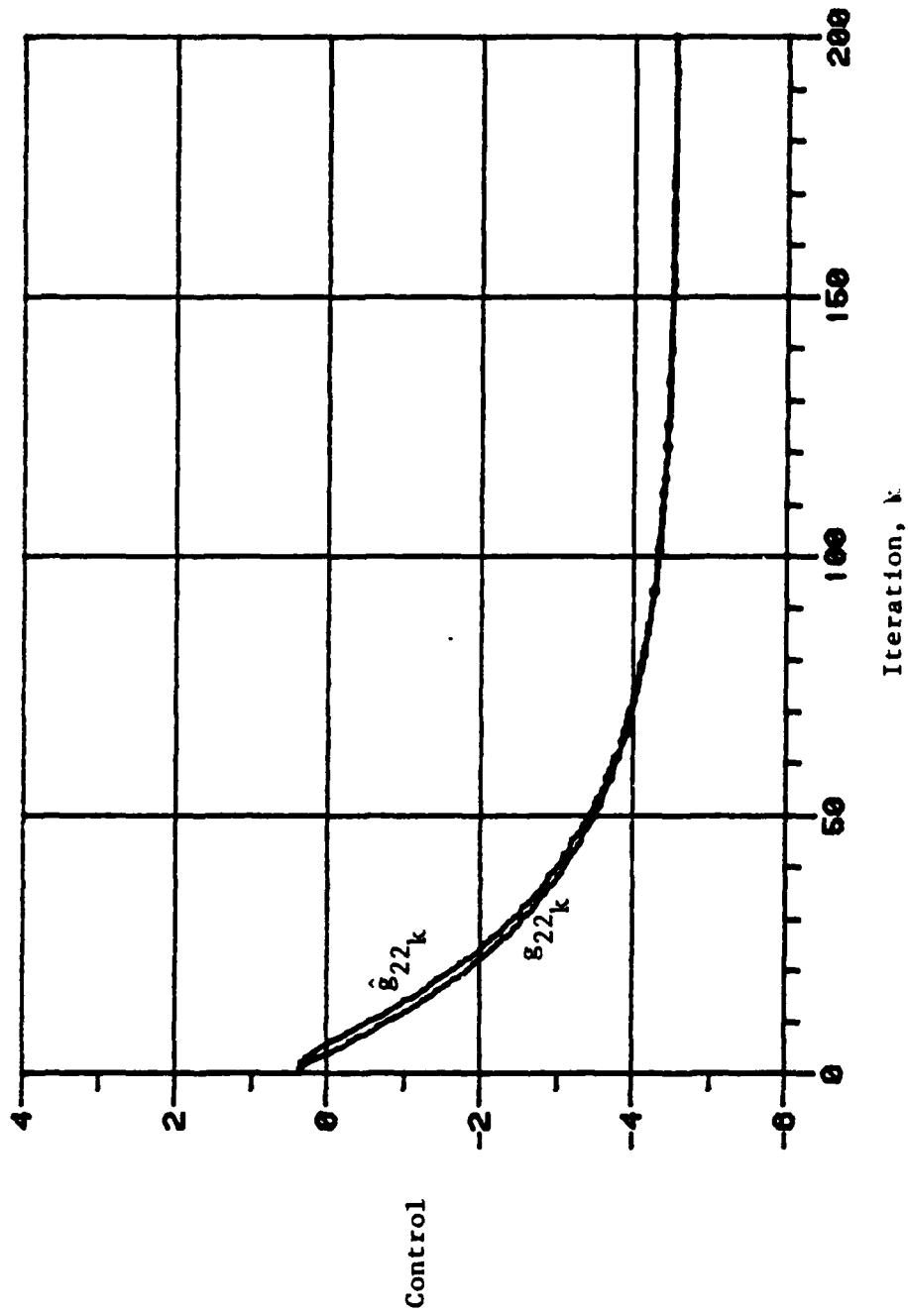
observe from Figures 12-15 that the noise affects the estimates $\hat{G}_{i_k}$ more severely than the actual controls G_{i_k} . The controls tend to their equilibrium values, but the estimates vary about the true control.

The estimates are more sensitive to noise than the actual inputs because the estimates are driven directly by the noise. A DM bases his estimates on his calculation of the other's previous input. In the noiseless case, he can determine the other DM's previous input exactly; however, when there is noise the measurement is corrupted, and he can only estimate the previous control. The functions (2.9)-(2.12) which determine the actual control act as a filter for the noisy estimates.

7. CONCLUSION

A simple method for determining the Nash equilibrium from the reaction relations of the decision makers has been presented.

An equilibrium was proposed and shown to be equivalent to the Nash equilibrium. The class of algorithms, which update based upon the error in the estimated state, was considered and it was proven that these algorithms could not converge to an incorrect value. A sample from this class was described and an example worked.

This work leaves open many areas for future work in the study of Nash games. For example, algorithms which have better convergence properties can be studied. Also, problems with different information structures should be investigated.

APPENDIX

PROOF OF NASH EQUILIBRIUM – THE CONSTANT K_i

The constant K_i is given by the following expression:

$$K_i = X_k'A'Q_iAX_k - 2X_k'A'Q_iC_i + C_i'Q_iC_i + 2X_k'A'Q_iB_1(F_{1_e}X_k + G_{1_e})$$

$$+ 2X_k'A'Q_iB_2(F_{2_e}X_k + G_{2_e}) + (F_{1_e}X_k + G_{1_e})'B_1'Q_iB_1(F_{1_e}X_k + G_{1_e})$$

$$+ (F_{2_e}X_k + G_{2_e})'B_2'Q_iB_2(F_{2_e}X_k + G_{2_e})$$

$$+ 2(F_{1_e}X_k + G_{1_e})'B_1'Q_iB_2(F_{2_e}X_k + G_{2_e})$$

$$+ (F_{i_e}X_k + G_{i_e})'R_i(F_{i_e}X_k + G_{i_e}) - 2C_i'Q_iB_1(F_{1_e}X_k + G_{1_e})$$

$$- 2C_i'Q_iB_2(F_{2_e}X_k + G_{2_e}).$$

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